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Long-term dispersion of contaminants in small estuaries

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It is shown that the use of axes moving with the tide (Shinohara *et al.* 1969) simplifies the analysis of contaminant dispersion in estuaries. Attention is restricted to estuaries which are small in the sense that cross-sectional mixing is rapid and that the tidal elevation can be taken to be constant along the estuary. In agreement with the work of Fischer (1972*a*, *b*) it is found that the dominant mechanism for dispersion is the transverse shear and not the vertical shear. Results are presented to illustrate the dependence of the upstream penetration of salt upon the estuary geometry as well as upon the fresh-water discharge rate.

1. Introduction

Much industrial development has been focused on estuaries. This has led to an increasing dependence upon estuaries for transport, recreation, drinking water, power-plant cooling and waste disposal. Some of these diverse uses can be directly affected, or can interfere with each other, when there are changes in the flow pattern or in the salinity distribution. For example, the 1976 drought in England and Wales was aggravated by the fact that many 'fresh-water' intakes could not be used because the salt water, and even sewage, had penetrated some tens of kilometres further upstream than was usual. An indication of the commercial importance of the long-term contaminant and salinity distributions in estuaries is given by the number and the large scale of the field studies and hydraulic and numerical models which have been reported in the open literature (Fischer 1976; Bowden 1967).

Analytical work on dispersion in estuaries has been hampered by the intrinsic complexity of nearly periodic turbulent stratified flows. There has been some success with semi-empirical model equations in which adjustable terms are introduced to compensate for the neglect of one or more of the following: transverse motion, vertical motion, tidal oscillations (Arons & Stommel 1951; Bowden 1965; Hansen & Rattray 1965; Harleman & Thatcher 1974). Recently, Imberger (1976) has shown that for buoyancy-driven flows in long shallow estuaries the reduced role of vertical stratification and the geometrical constraints imposed upon the flow make it possible to derive an analytic description of the dispersion. Here these geometrical ideas are applied to tidally dominated flows. An important feature of the present analysis is the use of axes convected with the tide, as advocated by Shinohara *et al.* (1969). This makes it possible to allow for substantial tidal amplitudes.

The convected co-ordinates are introduced and some simple consequences of volume and contaminant conservation are derived in § 2. Then in § 3 a preliminary simplification of the equations of motion is introduced, based upon the hypothesis that cross-sectional mixing takes place rapidly relative to the tidal period and that the slope of the free surface can be neglected (Smith 1976). A further simplification is made in §4 on the basis of the additional hypothesis that the estuary is much shallower than it is wide (Fischer 1967). The resulting description of the inter-tidal motion is used in §5 to find an expression for the dispersion coefficient (the effective longitudinal diffusivity). Finally, solutions for steady salinity distributions in estuaries of simple shapes are presented in §6.

The major hypotheses underlying the analysis are essentially geometric in nature. For example, with an average depth of 3 m and a transverse diffusivity of $0\cdot1\ m^2\,s^{-1}$ (Talbot & Talbot 1974), the above assumptions are satisfied if the estuary is much shorter than 200 km (the tidal wavelength), narrower than 200 m and much wider than 3 m. In practice the only serious restriction is the narrowness condition. Thus, before applying the results of this paper it should first be ascertained whether the main variation in the contaminant concentration occurs in a region for which the narrowness requirement is met. Of course, it is of little consequence if the modelling of dispersion should be inaccurate in a region where there is only a negligible concentration gradient. It must be emphasized that additional physical processes become important for wide estuaries (Holley, Harleman & Fischer 1970; Chatwin 1975). This will be the subject of a further paper.

As might be expected, the analysis provides new insight into several old results. For example, the tidal-averaged dispersion equation (Arons & Stommel 1951) is found to be more appropriate to axes moving with the tide than to the usual stationary axes. The gross effect of buoyancy can be quantified in terms of a single dimensionless number (Fischer 1976). Transverse shear is found to be the dominant mechanism for dispersion (Fischer 1967). Also, the dispersion coefficient for an oscillatory flow can be well approximated by that of a steady flow with the same mean absolute velocity (Bowden 1965; Holley & Harleman 1965) provided that the salinity gradient and the channel curvature are both small. For the last two results it is important to recall the requirement that cross-sectional mixing takes place rapidly (Holley *et al.* 1970).

2. Conservation laws in moving axes

To a first approximation the tide merely shifts the contaminant distribution back and forth along the estuary without any net dispersion. This has two unfortunate consequences as regards calculation procedures which use the usual fixed co-ordinates. First, most of the effort is expended watching the relatively rapid deformations of the flow. Second, it is essential that the advection be very accurately represented as otherwise spurious numerical diffusion can result (Harleman & Thatcher 1974). The most general method for avoiding these difficulties is to employ a fully Lagrangian formulation which fixes attention upon the motion of identified fluid elements (Fischer 1972*a*). However, for flows which are primarily unidirectional it suffices to use axes moving at the cross-sectional average longitudinal tidal velocity (Shinohara *et al.* 1969).

If x is the usual Eulerian distance along the centre-line of the estuary and t is the time, then we put

$$\int_{x_H}^{x} A(x',t)\,dx' = \int_{\xi_H}^{\xi} A(\xi',0)\,d\xi', \quad t=\tau. \tag{1a}$$

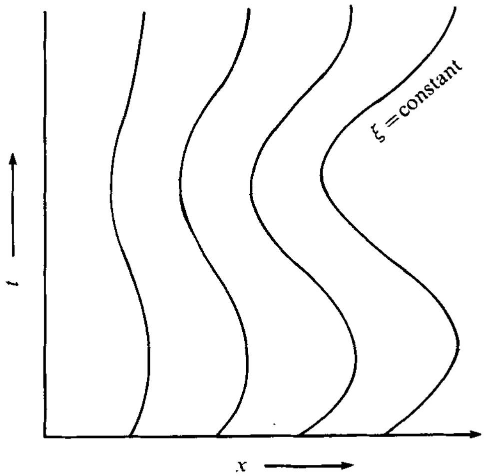

FIGURE 1. Concertina-like deformations of the contaminant distribution.

Here A is the cross-sectional area, $x_H(t)$ is the head of the estuary and ξ is the pseudo-Lagrangian co-ordinate (see figure 1). A routine calculation yields the transformation rules

$$\partial_x = (A/A_0)\,\partial_\xi, \quad \partial_t = \partial_\tau - U(A/A_0)\,\partial_\xi, \tag{1b}$$

where A_0 is the cross-sectional area at the initial time $\tau = 0$ and the local longitudinal velocity U of the co-ordinate system satisfies the equations

$$\partial_\xi U = -A_0\,\partial_\tau A/A^2, \quad AU = 0 \quad \text{at} \quad \xi = \xi_H. \tag{1c}$$

In its familiar Eulerian form the equation for the global conservation of volume is

$$\partial_t A + \partial_x(A\bar{u}) = 0,$$

where u is the longitudinal velocity and overbars are used to denote a cross-sectional average value. In the moving axes this equation takes the even simpler form

$$\partial_\xi[A(\bar{u} - U)] = 0. \tag{2a}$$

On performing the integration with respect to ξ, we can write

$$A(\bar{u} - U) = Q_H(\tau), \tag{2b}$$

where Q_H can be interpreted as the discharge of fresh water into the estuary. For tidally dominated flows Q_H is composed of two nearly cancelling terms.

If the longitudinal turbulent diffusion can be neglected then it is possible to derive a global conservation law for the concentration c of a conserved pollutant (such as salt):

$$\partial_t(A\bar{c}) + \partial_x(A\overline{uc}) = 0.$$

The analogous equation in the moving axes can be written

$$A_0\,\partial_\tau \bar{c} + Q_H\,\partial_\xi \bar{c} + \partial_\xi[A\overline{u(c-\bar{c})}] = 0.$$

For long-term dispersion the concentration can be assumed to be fairly well mixed across the estuary, i.e. $c - \bar{c}$ is small. This, together with the assumption that Q_H is small, enables us to infer that in the first approximation

$$\partial_\tau \bar{c} = 0. \tag{3a}$$

Thus, as has already been anticipated, on the tidal time scale there is no dispersion and the contaminant distribution is carried back and forth along the estuary.

In the next approximation we attempt to determine the slow evolution of the contaminant distribution. This implicit distinction between effects on the tidal time scale and effects on longer time scales can be formalized mathematically by the use of two time scales τ and T and by putting

$$\partial_\tau|_\xi = \partial_\tau|_{\xi,T} + \partial_T|_{\xi,\tau}$$

(Cole 1968, chap. 3). The short time scale τ is associated with the relatively rapid (semidiurnal or diurnal) variations in the phase of the major tidal constituents and, to avoid a profusion of long time scales, T is associated with all of the following: the dispersion of the contaminant, the time dependence of Q_H, and the frequency-splitting of the tidal constituents (or equivalently the variation in tidal amplitudes). Using angle brackets to denote an average with respect to the short time scale τ, we can deduce that on the long time scale $\bar{c}$ satisfies the longitudinal dispersion equation

$$A_0 \partial_T \bar{c} + Q_H \partial_\xi \bar{c} + \partial_\xi \langle A\overline{u(c-\bar{c})}\rangle = 0. \tag{3b}$$

Although the notation for the two types of averaging follows the usage of Fischer (1972*b*), here the interpretation of $\langle \ldots \rangle$ is slightly different owing to our use of tide-following axes.

To evaluate the final term in (3*b*) we must determine the cross-sectional structure both of the flow and of the concentration. In §5 it is found that for small shallow estuaries

$$\langle A\overline{u(c-\bar{c})}\rangle \doteq -A_0\langle (A/A_0)^2 E\rangle \partial_\xi \bar{c},$$

where E is the instantaneous longitudinal dispersion coefficient. Thus the dispersion equation has the familiar diffusion-equation structure (Taylor 1953):

$$A_0 \partial_T \bar{c} + Q_H \partial_\xi \bar{c} - \partial_\xi (A_0 \langle (A/A_0)^2 E\rangle \partial_\xi \bar{c}) = 0. \tag{3c}$$

This is equivalent to the equation by Shinohara *et al.* (1969). However, their modelling of the shear-dispersion term was empirical rather than deductive.

The boundary conditions at the ends of the estuary would typically be

$$Q_H \bar{c} - A_0\langle (A/A_0)^2 E\rangle \partial_\xi \bar{c} = Q_H c_H \quad \text{at} \quad \xi = \xi_H, \quad \bar{c} = c_M \quad \text{at} \quad \xi = \xi_M. \tag{3d}$$

Here c_H is the contaminant concentration of the inflow, c_M is the concentration far out at sea and ξ_M denotes the mouth of the estuary. For non-conservative contaminants, such as heat and domestic waste, extra terms would have to be included in (3*c*). Outflows situated at fixed Eulerian positions along the estuary would correspond to distributed source terms in the pseudo-Lagrangian equations (2*a*) and (3*c*).

The similarity between the co-ordinates ξ and x explains why reasonable predictions of contaminant dispersion can be obtained with the tidal-time-averaged Eulerian equation

$$\langle A\rangle \partial_t \bar{c} + Q_H \partial_x \bar{c} - \partial_x(\langle A\rangle \langle E\rangle \partial_x \bar{c}) = 0$$

(Arons & Stommel 1951). Indeed, a sufficient condition for this equation to be valid in the moving frame of reference is that the tidal variations in area are small. To justify the use of the equation in the usual fixed axes, we need the more stringent assumption that the tidal excursions are short relative to the length scales both of the contaminant distribution and of the channel geometry.

3. Equations of motion

The primary purpose of this and the next section is to obtain a sufficiently detailed mathematical description of the fluid and contaminant motion to permit the evaluation of the shear-dispersion term in (3*b*). Such is the complexity of the situation which we are studying that it is necessary to make many simplifying assumptions. For example, use has already been made of the assumptions that the longitudinal current is predominantly tidal, that the contaminant distribution is well mixed across the estuary, and that the motion can be characterized in terms of long and short time scales. In this section we develop the consequences of the estuary having relatively rapid lateral mixing and we make explicit the quantitative nature of the earlier assumptions. Consideration of the implications as regards dispersion are recommenced in §5.

The mathematical analysis given below has several points of contact with the work of Imberger (1976) and of Smith (1976) concerning the dispersion of buoyant contaminants. In particular, the estuary shape is assumed to be both narrow and shallow, turbulent mixing is represented by eddy diffusivities and we use a strong form of the Boussinesq approximation (i.e. gravity is such a strong force that the motion along the estuary can be driven by a negligibly small slope of the free surface). The most apparent differences from the work of Imberger (1976) are that here it is the longitudinal buoyancy-driven current that is neglected and not the tidal current, and also that Imberger foreshortens his analysis by combining the two levels of approximation described in this and in the next section. The major differences from the earlier work of the author (Smith 1976) are that here the longitudinal extent of the contaminant distribution and the oscillatory longitudinal velocity are an order of magnitude greater than was previously assumed. Thus longitudinal turbulent diffusion is of minor importance; the time scale for longitudinal dispersion greatly exceeds the tidal period; the contaminant distribution is advected a substantial distance back and forth with the tide, and the longitudinal buoyancy-driven current is relatively insignificant.

The hypothesis of rapid mixing is tantamount to assuming that the estuary is very narrow relative to its length. Thus we introduce an expansion parameter ϵ:

$$\epsilon^2 = \mathscr{B}/\mathscr{L},$$

where $\mathscr{B}$ and $\mathscr{L}$ are respectively typical width and length scales of the estuary. In order to perform a systematic analysis of the equations of motion we must specify the ϵ-ordering of the many terms relative to the basic dimensional quantities $\mathscr{U}_H$ and $\mathscr{B}$, where $\mathscr{U}_H$ is a typical discharge velocity associated with the fresh water. For any specific physical case an appropriate scaling can be ascertained from field or laboratory measurements (Imberger 1976). However, this can lead to unnecessarily restrictive assumptions. Here we make our results as widely applicable as possible by retaining as many physical effects as are compatible with the basic assumptions. The tidal elevation, lateral velocities and eddy diffusivities are of order ϵ^0; the tidal period, longitudinal velocity and the reduced gravity associated with salinity are of order ϵ^{-1}, and the evolution time scale is of order ϵ^{-2}.

To derive the 'maximum-generality' scalings the equations of motion were solved in the first instance for arbitrary scalings with the sole hypothesis of rapid mixing. Indeed, the initial tentative definition of the expansion parameter and the choice of basic dimensional quantities differed from those given above. Then more and more physical effects were assumed to have a leading-order effect upon the dispersion coefficient until the scalings became fully determined. The effects included at leading order are the longitudinal tidal current, buoyancy-driven transverse currents and Coriolis and centrifugal accelerations. However, it is assumed that the buoyancy-driven longitudinal current does not affect the dispersion at leading order. This is an opposite extreme to that studied by Imberger (1976) and by Smith (1976). The particular choice of $\mathscr{U}_H$ and $\mathscr{B}$ as basic quantities and the definition of the expansion parameter ϵ were made *a posteriori* to emphasize the mathematical similarities with the author's earlier work (Smith 1976).

The resulting version of the field equations and boundary conditions in the moving axes is

$$\epsilon^2\partial_T c+\epsilon\partial_\tau c+\epsilon(A/A_0)(u-U)\partial_\xi c+v\partial_y c+w\partial_z c = \partial_y(\kappa_2\partial_y c)+\partial_z(\kappa_3\partial_z c)+\epsilon^2R^{-1}\kappa_2\partial_y c+O(\epsilon^3), \quad (4a)$$

$$v\partial_y u+w\partial_z u+(A/A_0)\partial_\xi p = \partial_y(\nu_{12}\partial_y u)+\partial_z(\nu_{13}\partial_z u)+O(\epsilon), \quad (4b)$$

$$v\partial_y v+w\partial_z v-u^2R^{-1}+fu+\epsilon^{-3}\partial_y p = \partial_y(2\nu_{22}\partial_y v)+\partial_z[\nu_{23}(\partial_x v+\partial_y w)]+O(\epsilon), \quad (4c)$$

$$v\partial_y w+w\partial_z w+\epsilon^{-3}\partial_z p+\epsilon^{-1}\alpha g s = \partial_y[\nu_{23}(\partial_z v+\partial_y w)]+\partial_z(2\nu_{33}\partial_z w)+O(\epsilon), \quad (4d)$$

$$\epsilon(A/A_0)\partial_\xi u+\partial_y[(1+\epsilon^{-2}R^{-1}y)v]+(1+\epsilon^{-2}R^{-1}y)\partial_z w = 0, \quad (4e)$$

$$u = v = w = \partial_z c+\partial_y h\partial_y c+O(\epsilon^4) = 0 \quad \text{on} \quad z=-h, \quad (4f)$$

$$\partial_z u = \partial_z v+\partial_y w = \partial_z c = w-\epsilon\partial_\tau\zeta-\epsilon^2\partial_T\zeta = O(\epsilon^3) \quad \text{on} \quad z=\zeta. \quad (4g)$$

In these equations $(\epsilon^{-1}u, v, w)$ is the velocity vector, $\epsilon^{-2}R$ the radius of curvature of the centre-line of the estuary, ϵf the Coriolis parameter, $\epsilon^{-3}p$ the excess pressure above fresh-water hydrostatic, s the salinity, $\epsilon^{-1}\alpha g$ the reduced gravity, ζ the surface elevation, the κ_i are eddy diffusivities for the contaminant and the ν_{ij} are eddy diffusivities for momentum. The occurrence of the salinity in (4*d*) is based upon the assumption that salt is the only contaminant which results in long-term density variations sufficient to modify the flow. The significant differences from equations (1*a*–*g*) of Smith (1976) are the occurrence of τ, f and R and the change of scaling for p.

To simplify (4*a*–*g*) we employ regular perturbation expansions in powers of ϵ:

$$p = p^{(0)}+\epsilon p^{(1)}+\epsilon^2 p^{(2)}+\ldots, \quad \text{etc.},$$

where the $p^{(j)}$, etc., are all independent of ϵ. By hypothesis $c^{(0)}$ and $s^{(0)}$ are functions only of ξ and T (i.e. to the first approximation the contaminant distribution is laterally well mixed and is advected with the tide). Other immediate deductions are that the leading-order pressure is independent of y and z and that the lateral velocities can be represented in terms of a stream function:

$$v^{(0)} = \partial_z\psi, \quad w^{(0)} = -\partial_y\psi.$$

These results permit the leading-order non-trivial terms from (4*a*–*g*) to be combined into the much simpler equations

$$\partial_z\psi\partial_y c^{(1)}-\partial_y\psi\partial_z c^{(1)}-\partial_y(\kappa_2\partial_y c^{(1)})-\partial_z(\kappa_3\partial_z c^{(1)}) = (U^{(0)}-u^{(0)})(A/A_0)\partial_\xi c^{(0)}, \quad (5a)$$

$$\partial_z\psi\partial_y u^{(0)}-\partial_y\psi\partial_z u^{(0)}-\partial_y(\nu_{12}\partial_y u^{(0)})-\partial_z(\nu_{13}\partial_z u^{(0)}) = U^{(0)}F, \quad (5b)$$

$$(\partial_y \psi \partial_z - \partial_z \psi \partial_y)[\partial_y^2 \psi + \partial_z^2 \psi] + (\partial_z^2 - \partial_y^2)[\nu_{23}(\partial_z^2 - \partial_y^2)\psi] + 2\partial_y \partial_z [(\nu_{22} + \nu_{33})\partial_y \partial_z \psi]$$
$$= -\alpha g \partial_y s^{(1)} - 2R^{-1} u^{(0)} \partial_z u^{(0)} + f \partial_z u^{(0)}, \quad (5c)$$

$$u^{(0)} = \psi = \partial_z \psi + \partial_y h \partial_y \psi = \partial_z c^{(1)} + \partial_y h \partial_y c^{(1)} = 0 \quad \text{on} \quad z = -h, \quad (5d)$$

$$\partial_z u^{(0)} = \psi = \partial_z^2 \psi = \partial_z c^{(1)} = 0 \quad \text{on} \quad z = \zeta. \quad (5e)$$

Without loss of generality we require that $c^{(1)}$ has zero cross-sectional average and that the longitudinal pressure gradient $U^{(0)}F(\xi, \tau, T)$ adjusts to achieve the required value of $\bar{u}^{(0)}$.

The results in §2 were derived without reference to an explicit approximation scheme. For consistency, we now note the formal equivalents of the major results:

$$\left.\begin{aligned} \partial_\xi U^{(0)} &= -A_0 \partial_\tau A/A^2, \quad AU^{(0)} = 0 \quad \text{at} \quad \xi = \xi_H, \\ \partial_\xi U^{(1)} &= -A_0 \partial_T A/A^2, \quad AU^{(1)} = 0 \quad \text{at} \quad \xi = \xi_H, \end{aligned}\right\} \quad (1c^*)$$

$$\bar{u}^{(0)} = U^{(0)}, \quad A(\bar{u}^{(1)} - U^{(1)}) = Q_H(T), \quad (2b^*)$$

$$A_0 \partial_T c^{(0)} + Q_H \partial_\xi c^{(0)} + \partial_\xi \langle A \overline{u^{(0)} c^{(1)}} \rangle = 0. \quad (3b^*)$$

Equations $(2b^*)$ can be derived directly by integrating the order-ϵ and order-ϵ^2 terms in the continuity equation $(4e)$ across the estuary. Similarly, the order-ϵ^2 terms in the diffusion equation $(4a)$ permit a direct derivation of $(3b^*)$.

4. Shallow-water expansion

The equations $(5a–e)$ for the inter-tidal motion are still intractably complicated. One possible simplification is to use series expansions based upon the assumption that the effects of buoyancy, curvature and rotation are small (Erdogen & Chatwin 1967). Instead, we shall follow Fischer (1967) and base an approximation scheme on the fact that estuaries are typically much shallower than they are wide. Thus we introduce a second small parameter

$$\delta = \mathscr{H}/\mathscr{B}$$

and define a new vertical co-ordinate

$$z^* = \delta^{-1} z,$$

where $\mathscr{H}$ is a typical channel depth. When ϵ and δ are related it is possible, as is done by Imberger (1976), to incorporate this second stage of approximation in that of §3.

In order to retain the maximum number of physical effects and, secondarily, to give equal importance to all of the terms in the dispersion equation $(3b^*)$, we are led to specify the scalings

$$\nu_{ij} = \delta \nu_{ij}^*, \quad \kappa_{ij} = \delta \kappa_{ij}^*, \quad \psi = \delta \psi^*, \quad \zeta = \delta \zeta^*, \quad \alpha = \delta^{-2} \alpha^*,$$
$$F = \delta^{-1} F^*, \quad f = \delta^{-\frac{1}{2}} f^*, \quad u = \delta^{-\frac{1}{2}} u^*, \quad c^{(1)} = \delta^{\frac{1}{2}} c^{(1)*},$$

where the starred quantities are of order 1. These scalings were derived on a similar basis to those in the previous section. The considerable difference in scalings from the work of Smith (1976) is due both to a different choice of basic dimensional quantities and to the retention of different physical effects.

It is not to be expected that the maximum-generality scalings provide the most effective way of viewing all estuaries. We need to produce specifications for $\mathscr{L}$, $\mathscr{B}$, $\mathscr{H}$

and $\mathscr{U}_H$ which imply realistic scalings for the many other physical terms. Otherwise the analysis may be regarded as being over-elaborate. Let us take the case of the River Colne downstream of Colchester (England) with an above average (but by no means exceptional) flow rate. It may be characterized by

$$\mathscr{L} = 10\,\text{km}, \quad \mathscr{B} = 100\,\text{m}, \quad \mathscr{H} = 4\,\text{m}, \quad \mathscr{U}_H = 0{\cdot}04\,\text{m}\,\text{s}^{-1}.\dagger$$

The maximum-generality scalings for the other physical quantities are

tidal elevation, 4 m; lateral velocities, 0·04 m s^{-1}; eddy diffusivities, 0·2 $\text{m}^2\,\text{s}^{-1}$; Coriolis parameter, 2×10^{-4} s; longitudinal velocity, 2 m s^{-1}; reduced gravity, 0·1 m s^{-2}.

None of these scalings are seriously unrealistic (Talbot & Talbot 1974). Thus, for the River Colne at least, it appears to be *a priori* necessary to utilize the full scope of the present analysis.

With the stars suppressed the new version of (5*a*–*e*) is

$$-\partial_z(\kappa_3\,\partial_z\,c^{(1)}) + \delta\{\partial_z(\psi\partial_y\,c^{(1)}) - \partial_y(\psi\partial_z\,c^{(1)})\} - \delta^2\partial_y(\kappa_2\,\partial_y\,c^{(1)}) = \delta^2(U^{(0)} - u^{(0)})\,(A/A_0)\,\partial_\xi\,c^{(0)},$$

$$-\partial_z(\nu_{13}\,\partial_z\,u^{(0)}) + O(\delta) = U^{(0)}F,$$

$$\partial_z^2(\nu_{23}\,\partial_z^2\psi) + O(\delta) = -\alpha g\partial_y\,s^{(1)} - 2R^{-1}u^{(0)}\partial_z\,u^{(0)} + f\partial_z\,u^{(0)},$$

$$u^{(0)} = \psi = \partial_z\,\psi + \delta^2\partial_y\,h\partial_y\,\psi = \partial_z\,c^{(1)} + \delta^2\partial_y\,h\partial_y\,c^{(1)} = 0 \quad \text{on} \quad z = -h,$$

$$\partial_z\,u^{(0)} = \psi = \partial_z^2\psi = \partial_z\,c^{(1)} = 0 \quad \text{on} \quad z = \zeta.$$

We now make explicit use of the fact that δ is small and formally seek regular perturbation solutions to the above equations:

$$c^{(1)} = c_0^{(1)} + \delta c_1^{(1)} + \delta^2 c_2^{(1)} + \dots .$$

After a straightforward but lengthy calculation, we find that if the eddy diffusivities are independent of z then

$$u_0^{(0)} = \tfrac{1}{2}U^{(0)}F[(\zeta+h)^2 - (\zeta-z)^2]/\nu_{13}, \tag{6a}$$

$$F = 3A\bigg/\int_{y_-}^{y_+}(\zeta+h)^3\,\nu_{13}^{-1}\,dy, \tag{6b}$$

$$\begin{aligned}\psi_0 = &-\{2(\zeta-z)^4 - 3(\zeta-z)^3(\zeta+h) + (\zeta-z)\,(\zeta+h)^3\}\,\alpha g\partial_y\,s_0^{(1)}/48\nu_{23}\\ &+\{(\zeta-z)^7 - 7(\zeta-z)^5(\zeta+h)^2 + 11(\zeta-z)^3(\zeta+h)^4 - 5(\zeta-z)\,(\zeta+h)^6\}\,U^{(0)2}F^2/840R\nu_{13}^2\,\nu_{23}\\ &+\{(\zeta-z)^5 - 2(\zeta-z)^3(\zeta+h)^2 + (\zeta-z)\,(\zeta+h)^4\}\,U^{(0)}Ff/120\nu_{13}\,\nu_{23},\end{aligned} \tag{6c}$$

$$\partial_y\,c_0^{(1)}\left\{\kappa_2(+\zeta h) + \kappa_3^{-1}\int_{-h}^{\zeta}\psi_0^2\,dz\right\} = (A/A_0)\,\partial_\xi\,c^{(0)}\int_{y_-}^{y}dy'\int_{-h}^{\zeta}(u_0^{(0)} - U^{(0)})\,dz, \tag{6d}$$

where $y_\pm$ are the two sides of the channel. The restriction upon the form of the eddy diffusivities serves merely to shorten the length and complexity of the above solutions. The expression for ψ_0 shows that the transverse flow near the free surface responds to the buoyancy, centrifugal and Coriolis forces and that there is a return flow near the channel bed. Equation (6*d*) is a mathematical statement of the major mechanism

† *The Surface Water Year Book of Great Britain* 1966–1970. Water Resources Board & Scottish Development Department, 1974. H.M.S.O.

underlying longitudinal dispersion (Taylor 1953): that there is a balance between cross-sectional mixing as augmented by the cross-flow and the longitudinal spreading due to non-uniform advection.

On performing the vertical integration, (6*d*) can be rewritten as

$$\partial_y c_0^{(1)}\{K+[K_{RR}+K_{fR}+K_{ff}+K_{RS}+K_{fS}+K_{SS}]\} = Q_0(A/A_0)\,\partial_\xi c^{(0)}, \tag{7}$$

where we have used the abbreviated notation

$$\begin{aligned}
Q_0 &= \int_{y-}^{y} dy' \int_{-h}^{\zeta} (u_0^{(0)}-U^{(0)})\,dz = \int_{y+}^{y} dy' \int_{-h}^{\zeta} (u_0^{(0)}-U^{(0)})\,dz,\\
K &= \kappa_2(\zeta+h),\\
K_{RR} &= 37\,888\, U^{(0)4}F^4(\zeta+h)^{15}/45\,045(840)^2R^2\nu_{13}^4\,\nu_{23}^2\,\kappa_3,\\
K_{fR} &= -15\,872 fU^{(0)3}F^3(\zeta+h)^{13}/45\,045\,.\,120\,.\,840\,R\nu_{13}^3\,\nu_{23}^2\,\kappa_3,\\
K_{ff} &= 128 f^2U^{(0)2}F^2(\zeta+h)^{11}/3465(120)^2\,\nu_{13}^2\,\nu_{23}^2\,\kappa_3,\\
K_{RS} &= 1207\alpha g\partial_y s_0^{(1)}U^{(0)2}F^2(\zeta+h)^{12}/6930\,.\,48\,.\,840\,R\nu_{13}^2\,\nu_{23}^2\,\kappa_3,\\
K_{fS} &= -\alpha g\partial_y s_0^{(1)} fU^{(0)}F(\zeta+h)^{10}/15\,.\,48\,.\,120\,\nu_{13}\,\nu_{23}^2\,\kappa_3,\\
K_{SS} &= 19(\alpha g\partial_y s_0^{(1)})^2(\zeta+h)^9/630\,(48)^2\,\nu_{23}^2\,\kappa_3.
\end{aligned}$$

For the important special case in which salt is the contaminant, (7) is a cubic equation for $\partial_y s_0^{(1)}$. In extreme cases there can be three real roots, with jumps between the roots corresponding to front-like discontinuities. We note that Q_0 involves only the transverse shear and not the vertical shear.

In the above analysis it is implicit that $\epsilon \ll \delta \ll 1$. However, at the expense of lengthier calculations, the same results (6*a*–*d*) can be obtained with a double expansion

$$c = \sum_{i,j} \epsilon^i \delta^j c_j^{(i)}, \quad \epsilon \ll 1, \quad \delta \ll 1.$$

Thus the loss of generality is illusory.

5. Dispersion coefficient

We are now in a position to evaluate the shear-dispersion term in (3*b**). Making use of the facts that $c_0^{(1)}$ has zero cross-sectional average and is independent of z, we can show that

$$\begin{aligned}
\langle A\overline{u_0^{(0)}c_0^{(1)}}\rangle &= \langle A\overline{(u_0^{(0)}-U^{(0)})\,c_0^{(1)}}\rangle = -\langle A\overline{Q_0\,\partial_y c_0^{(1)}}\rangle\\
&= -A_0\langle (A/A_0)^2 E\rangle\,\partial_\xi c^{(0)},
\end{aligned} \tag{8a}$$

where E is the instantaneous dispersion coefficient

$$E = A^{-1}\int_{y-}^{y+} Q_0^2\,\{K+[K_{RR}+K_{fR}+K_{ff}+K_{RS}+K_{fS}+K_{SS}]\}^{-1}\,dy \tag{8b}$$

(Smith 1976). Thus the dispersion equation takes the form

$$A_0\,\partial_T c^{(0)} + Q_H\,\partial_\xi c^{(0)} - \partial_\xi(A_0\langle (A/A_0)^2E\rangle\,\partial_\xi c^{(0)}) = 0, \tag{3c*}$$

where $c^{(0)}$ may be interpreted either as the first approximation to the contaminant concentration or as the cross-sectional average concentration $\bar{c}$. In applications

($8b$) and ($3c^*$) can be used with unscaled, dimensional variables, the main requirement for validity being that cross-sectional mixing takes place in less than the tidal period.

We recall that Q_0 is an incomplete integral across the estuary of the vertically integrated volume flux, that K is the vertically integrated turbulent transverse diffusion coefficient and that the subscripted K terms show how centrifugal, Coriolis and transverse buoyancy effects augment the transverse diffusion. To evaluate the salinity terms it is first necessary to solve the cubic equation (7). With this identification of terms ($3b^*$) and ($8b$) clearly illustrate three well-known results: first, that for estuaries with a transverse-mixing time scale shorter than the tidal period the dispersion is dependent on the transverse shear and not the vertical shear (Holley *et al.* 1970); second, that for salt the dispersion equation ($3c^*$) is strongly nonlinear in that E is a function of the salinity gradient (Erdogan & Chatwin 1967; Chatwin 1976); third, that unless the contaminant actively modifies the turbulence the dispersion coefficient for neutrally buoyant contaminants is the same as that for salt (Stommel 1953).

Equation ($8b$) differs slightly from the results derived by Smith (1976). In the present analysis allowance is made for the curvature of the channel and the effect of the earth's rotation, hence the occurrence of the new transverse circulation terms K_{RR}, K_{fR}, K_{ff}, K_{RS} and K_{fS}. However, the buoyancy-driven longitudinal circulation has been ignored and the incomplete integral of the volume flux does *not* take the form

$$Q_0 + (A/A_0)\, \partial_\xi s^{(0)} Q_1.$$

This is a consequence of our having assumed that the longitudinal extents of the salinity distribution and the tidal current are an order of magnitude greater than was previously assumed. There is a very brief time interval close to the turn of the tide in which the Q_1 term is significant (Smith 1976). In the present context it is the tidal average ($8a$) that is of concern, and for this purpose ($8b$) is quite adequate.

The scaling assumptions made by Imberger (1976) focus attention on a restricted set of physical effects and his result for the dispersion coefficient includes only the Q_1 and K terms. Furthermore, in his analysis the estuary is assumed to have a rigid cover in contact with the water. Thus there is no possibility of direct comparison with the present work.

A particularly simple, and fairly realistic, way of modelling the variation of the eddy diffusivities with position and time is by means of the formulae

$$\kappa_i \doteq \kappa_i' \,|\hat{u}_0^{(0)}|\, (h+\zeta), \quad \nu_{ij} = \nu_{ij}' \,|\hat{u}_0^{(0)}|\, (h+\zeta), \tag{9}$$

where κ_i' and ν_{ij}' are empirical constants and $\hat{u}$ is the velocity at the free surface (Talbot & Talbot 1974). These relationships considerably simplify the dimensional structure of (6)–(8). In particular, the dispersion coefficient E scales as

$$EH/B^2 U_p, \tag{10a}$$

where $U_p(\xi)$ is the local peak value of $U^{(0)}(\xi, \tau)$ and H and B are the local counterparts to $\mathscr{H}$ and $\mathscr{B}$. Also, it is convenient to measure the importance of buoyancy, curvature and the earth's rotation in terms of the dimensionless groups

$$G = \alpha g H B (A/A_0)\, \partial_\xi s^{(0)}/U_p^2, \quad H/R, \quad fH/U_p. \tag{10b}$$

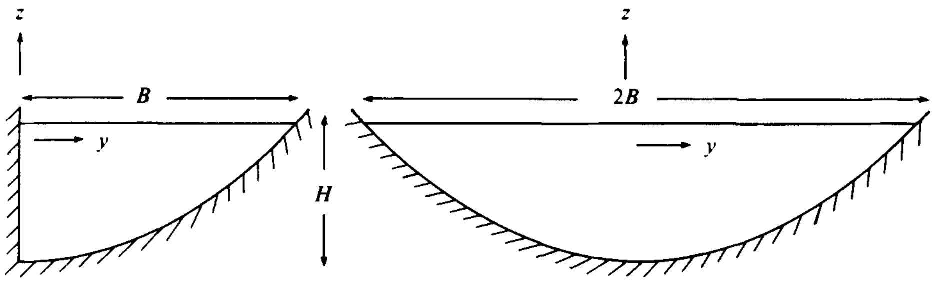

FIGURE 2. Definition sketch of the cross-sections used in the analysis.

From the dispersion equation ($3c^*$) we find that an order-of-magnitude estimate for G is given by the local 'estuarine Richardson number'

$$Ri = Q_H \alpha g H / B^2 U_p^3. \tag{10c}$$

This differs by the factor H/B from the definition employed by Fischer (1972*b*).

For a channel of parabolic cross-section (as shown in figure 2) and with

$$\kappa_3' = \nu_{23}' = 0{\cdot}005, \quad \kappa_2' = 0{\cdot}02,$$

the coefficients in (7) and (8*b*) have the values

$$Q_0 = BHU^{(0)}\left\{\frac{4}{3\pi}\sin^{-1}(y/B) + \frac{4}{9\pi}(y/B)\,[5-2(y/B)^2]\,[1-(y/B)^2]^{\frac{1}{2}} - (y/B)\,[1-\tfrac{1}{3}(y/B)^2]\right\},$$

$$K = 0{\cdot}03395\,H^2|U^{(0)}|\,[1-(y/B)^2]^{\frac{5}{2}},$$

$$K_{RR} = 259{\cdot}0\,(H/R)^2\,H^2|U^{(0)}|\,[1-(y/B)^2]^{\frac{9}{2}},$$

$$K_{fR} = -223{\cdot}7\,(H/R) fH^3 \operatorname{sgn}(U^{(0)})\,[1-(y/B)^2]^4,$$

$$K_{ff} = 48{\cdot}36 f^2H^2\,|U^{(0)}|^{-1}\,[1-(y/B)^2]^{\frac{7}{2}},$$

$$K_{RS} = 81{\cdot}42\,(H/R)\,\alpha g\,\partial_y s_0^{(1)} H^4\,|U^{(0)}|^{-1}\,[1-(y/B)^2]^{\frac{9}{2}},$$

$$K_{fS} = -64{\cdot}25\,\alpha g\,\partial_y s_0^{(1)} fH^5\,|U^{(0)}|^{-2} \operatorname{sgn}(U^{(0)})\,[1-(y/B)^2]^4,$$

$$K_{SS} = 21{\cdot}40\,(\alpha g\,\partial_y s_0^{(1)})^2 H^6 |U^{(0)}|^{-3}\,[1-(y/B)^2]^{\frac{9}{2}}.$$

The range of sizes of the numerical factors is due to our use of unscaled quantities. For example, it is implicit that H/R is of order $\epsilon\delta$. Figures 3(*a*), (*b*) and (*c*) show how buoyancy, curvature and the earth's rotation each in turn decrease the instantaneous dispersion coefficient.

We note that K is proportional to κ_2' and that the subscripted terms vary inversely as $\kappa_3' \nu_{23}'^2$. Thus the results are fairly sensitive to the choice of constants. In particular, for flows in which turbulence is the dominant transverse mixing mechanism, the dispersion coefficient E has a factor of minus one sensitivity with respect to the constant κ_2'. The position in the parameter space (10*b*) of the transition between turbulence-dominated and circulation-dominated mixing has a factor of two sensitivity, while for flows dominated by Coriolis, curvature or buoyancy effects E has a sensitivity of plus three.

In evaluating the tidal average (8*a*) care must be taken to allow for the variations in breadth, depth and channel curvature as the axes are advected up and down the

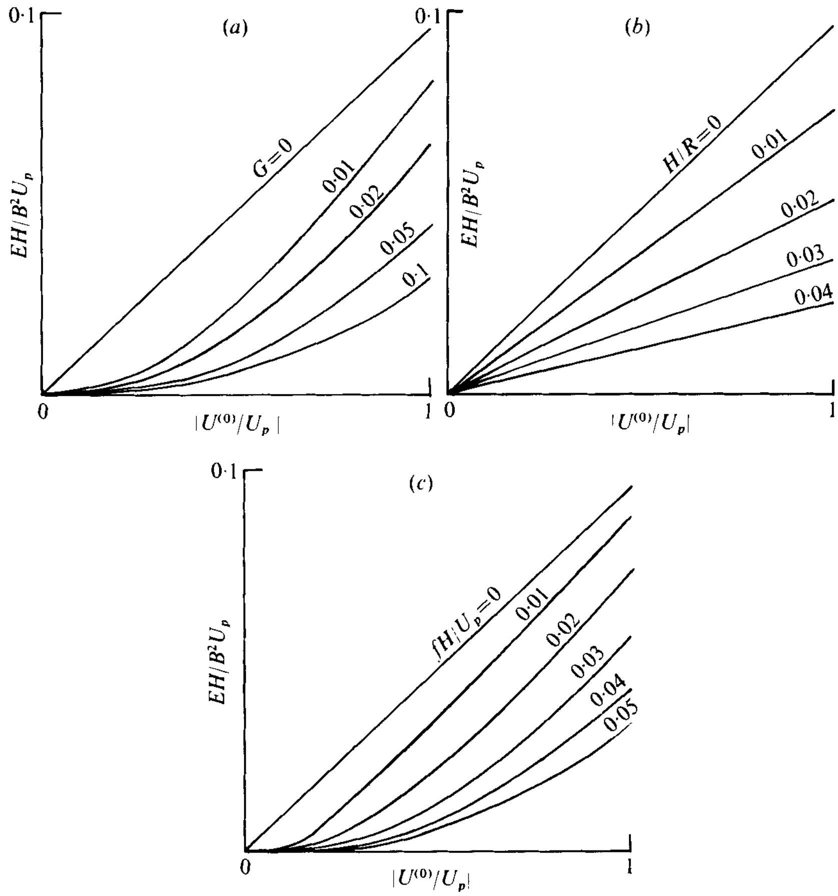

FIGURE 3. The dependence of the instantaneous dispersion coefficient E upon (a) the salinity gradient, (b) the channel curvature and (c) the Coriolis effect.

estuary. If such variations are small, then the linearity of figures 3 (a)–(c) indicates the efficacy of the rule of thumb (Bowden 1965; Holley & Harleman 1965): that the dispersion coefficient is the same as if the flow were steady with velocity equal to the mean absolute velocity of the oscillatory flow. Again, we recall the important proviso that the period of oscillation is longer than the time scale for cross-sectional mixing (Holley *et al.* 1970). If the effects of buoyancy, curvature and rotation are neglected then for parabolic channels

$$E = 0{\cdot}096\, B^2 |U^{(0)}|/H. \tag{11}$$

The consistently small value of EH/B^2U_p means that the estimate (10 c) for G is low by at least a factor of ten.

Having done all the above computations we can now re-assess the importance of the various physical effects in the case of the River Colne. Thus we specify

$$R = 400\,\text{m}, \quad B = 50\,\text{m}, \quad H = 4\,\text{m}, \quad f = 10^{-4}\,\text{s}^{-1},$$
$$U_p = 0{\cdot}7\,\text{m}\,\text{s}^{-1}, \quad Q_H = 5\,\text{m}^3\,\text{s}^{-1}.$$

With allowance for a factor of ten in G, this leads to the estimates

$$G = 0{\cdot}06, \quad H/R = 0{\cdot}01, \quad fH/U_p = 0{\cdot}0006.$$

Thus, from figures 3(*a*)–(*c*) we can infer that it is reasonable to neglect the Coriolis effect, but that the salinity gradient and the channel curvature can greatly reduce the dispersion. To get agreement with Talbot & Talbot's (1974) measured longitudinal dispersion coefficient for the River Colne of $14{\cdot}2\,\mathrm{m^2\,s^{-1}}$ we need take the tidal average of EH/B^2U_p to be 0·03. This is reassuringly close to our estimates (see figure 3*a*).

6. Simple solutions

Along a real estuary the channel depth and breadth vary markedly. Thus, unless we can approximate these variations by simple functions (Shinohara *et al.* 1969), the transformation (1*a*) to tide-following axes will have to be done numerically. Furthermore, if buoyancy, curvature or Coriolis effects are significant then analytic expressions, such as (11), are not available and it will be necessary to calculate E numerically. For some purposes qualitative results are sufficient. For example, to what extent can field results in one estuary be used to make predictions about another estuary? One means of addressing such questions is via special solutions for a class of simplified problems. It is to this end that this section is directed.

The first and simplest example concerns a uniformly descending valley of parabolic cross-section (figure 4*a*). The analogous situation for a triangular cross-section was analysed by Shinohara *et al.* (1969). In Eulerian co-ordinates the profile is specified as

$$z/\mathscr{H} = -(x/\mathscr{L}) + (y/\mathscr{B})^2,$$

where $\mathscr{H}/\mathscr{L}$ is the very small slope of the valley bottom. Since the tidal elevation ζ is assumed to be constant along the estuary, there is merely a displacement of the entire water region

$$x = \xi - \zeta(\mathscr{L}/\mathscr{H}), \quad U^{(0)} = -\partial_\tau \zeta(\mathscr{L}/\mathscr{H}), \tag{12a}$$

where we have chosen the time origin to correspond to $\zeta = 0$.

The local channel depth and breadth scales are given by

$$H = \mathscr{H}(\xi/\mathscr{L}), \quad B = \mathscr{B}(\xi/\mathscr{L})^{\frac{1}{2}}.$$

It is these ξ-dependent scalings which we must use in the results derived in the previous section. In particular, if buoyancy and Coriolis effects are negligible, then from (11) we find

$$\langle (A/A_0)^2 E\rangle = 0{\cdot}096(a\Omega\mathscr{B}^2\mathscr{L}/\mathscr{H}^2)\langle|\partial_\tau \zeta/a\Omega|\rangle, \tag{12b}$$

where Ω is the tidal frequency and a is a typical tidal amplitude.

In order to non-dimensionalize the longitudinal dispersion equation (3*c**) we define

$$\tilde{\xi} = \xi/\mathscr{L}, \quad \tilde{T} = T\Omega a\mathscr{B}^2/\mathscr{H}^2\mathscr{L}, \quad \tilde{\mathscr{U}}_H = \tfrac{4}{3}Q_H\mathscr{H}/\mathscr{B}^3\Omega a, \quad \mu = 0{\cdot}096\langle|\partial_\tau\zeta/a\Omega|\rangle. \tag{13}$$

Thus $\mathscr{U}_H$ is a dimensionless discharge velocity associated with the fresh water. Suppressing the tildes, the resulting equation is

$$\xi^{\frac{3}{2}}\partial_T \bar{c} + \mathscr{U}_H\,\partial_\xi \bar{c} - \mu\partial_\xi(\xi^{\frac{3}{2}}\partial_\xi \bar{c}) = 0, \tag{12c}$$

where we have identified $c^{(0)}$ with the cross-sectional average concentration $\bar{c}$.

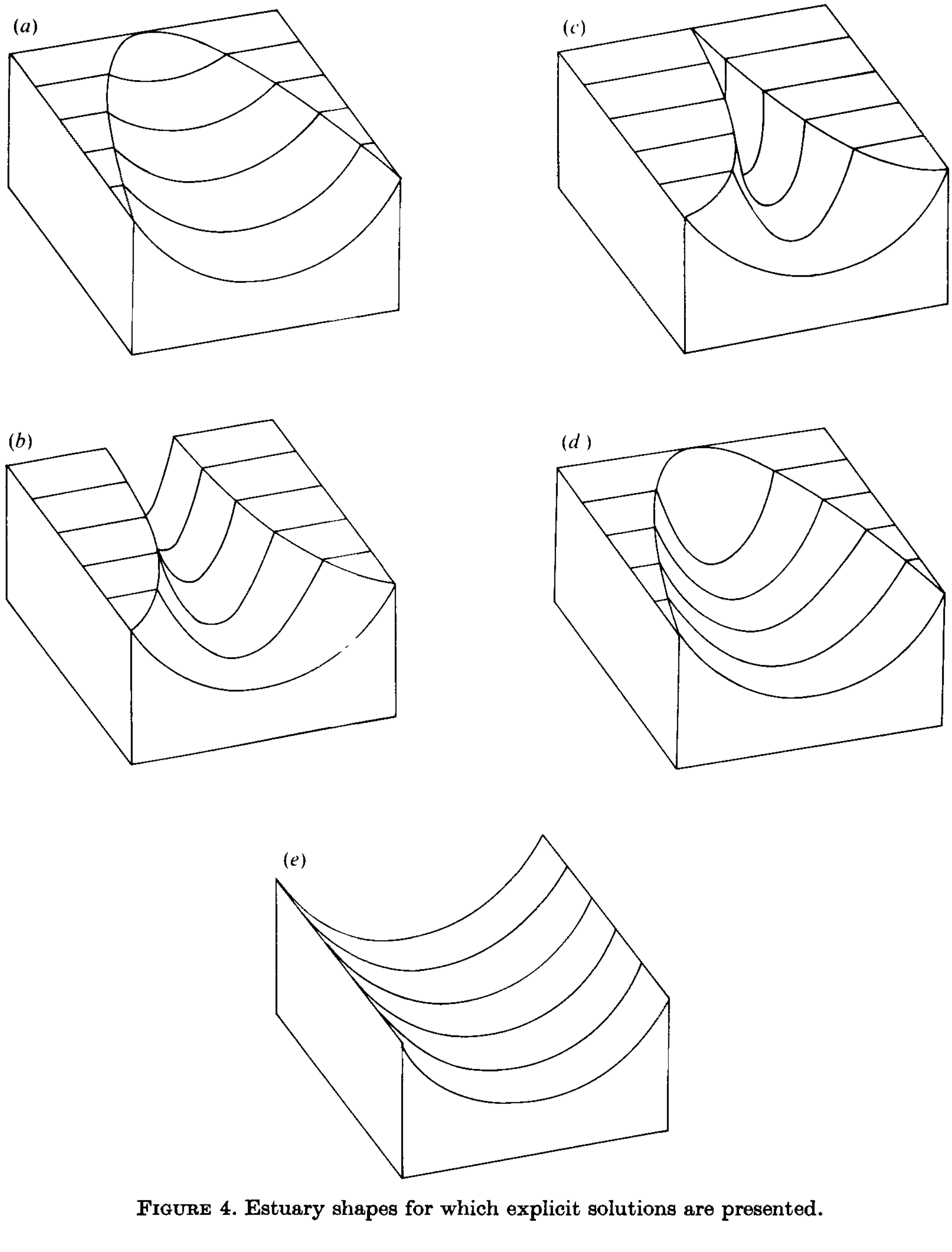

FIGURE 4. Estuary shapes for which explicit solutions are presented.

If the flow conditions are independent of time, then (12*c*), together with the boundary conditions (3*d*), has the steady solution

$$\bar{c} = c_H + (c_M - c_H)\exp\{2\mu^{-1}\mathscr{U}_H(\xi_M^{-\frac{1}{2}} - \xi^{-\frac{1}{2}})\}. \tag{12d}$$

Thus, for small $\mathscr{U}_H$ the transition between upstream and sea conditions takes place on the length scale $4\mu^{-2}\mathscr{U}_H^2\,\mathscr{L}$. This implies that in drought conditions the upstream

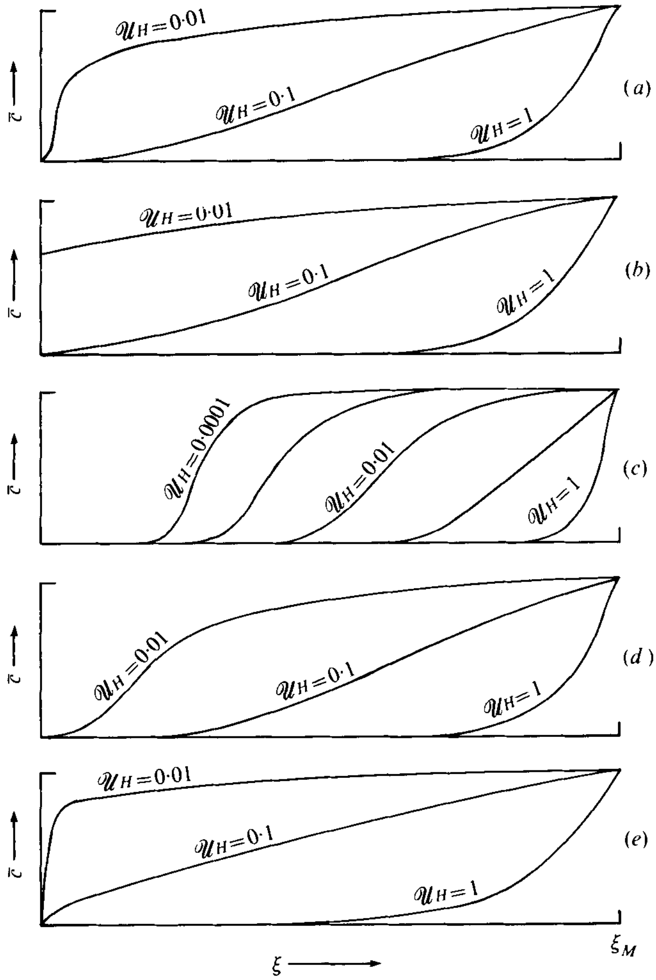

FIGURE 5. Steady contaminant distributions corresponding to several values of the non-dimensional fresh-water discharge velocity for the estuary shapes shown in figure 4.

penetration of salt or other contaminants becomes very sensitive to the fresh-water discharge. Also, when we test whether the estuary meets the narrowness requirement, the most appropriate width for us to use is $2\mu^{-1}\mathscr{U}_H\mathscr{B}$. This is because further out to sea the concentration gradients are small and any unreliability of the estimate of E, due to the estuary being wide, does not significantly affect the overall solution for $\bar{c}$.

Figure 5(*a*) shows the result (12*d*) for the special case

$$c_H = 0, \quad c_M = 1, \quad \xi_M = 1, \quad \langle|\partial_\tau \zeta/a\Omega|\rangle = 1.$$

It is only for very small $\mathscr{U}_H$ that the profiles have the S-shape characteristic of salinity distributions in estuaries (Arons & Stommel 1951). However, realistic estimates of the fresh-water discharge rate Q_H do indeed yield appropriately small values for $\mathscr{U}_H$. For example, the value $\mathscr{U}_H = 0{\cdot}01$ corresponds to

$$Q_H = 25\,\mathrm{m^3\,s^{-1}}, \quad \mathscr{H} = 10\,\mathrm{m}, \quad \mathscr{B} = 500\,\mathrm{m}, \quad a = 1\,\mathrm{m}.$$

If the effect of the salinity gradient is not negligible, then there is a reduction in the longitudinal dispersion coefficient (see figure 3*a*). Qualitatively we can infer that there will be a compensating steepening of the central portion of the salinity profiles, making the S-shape more pronounced.

The second example concerns an exponentially widening valley of parabolic cross-section (see figure 4*b*). The Eulerian specification is

$$z/\mathscr{H} = -1 + (y/\mathscr{B})^2 \exp(-2x/\mathscr{L}), \quad -\infty < x < x_M.$$

When the surface elevation is ζ, the volume of water upstream of a position x is given by

$$\tfrac{4}{3}\mathscr{B}\mathscr{H}\mathscr{L}[1+\zeta/\mathscr{H}]^{\frac{3}{2}} \exp(x/\mathscr{L}).$$

Now, the volume upstream of a position ξ remains constant when we use axes moving with the tide. Thus we can deduce that

$$x/\mathscr{L} = \xi/\mathscr{L} - \tfrac{3}{2}\ln[1+\zeta/\mathscr{H}], \quad U^{(0)} = -\tfrac{3}{2}(\mathscr{L}/\mathscr{H})[1+\zeta/\mathscr{H}]^{-1}\partial_\tau\zeta, \tag{14a}$$

where once again we have taken $\zeta = 0$ to define the time origin.

The local channel depth and breadth scales are given by

$$H = \mathscr{H}[1+\zeta/\mathscr{H}], \quad B = \mathscr{B}[1+\zeta/\mathscr{H}]^{-1}\exp(\xi/\mathscr{L}).$$

If the effect of the salinity gradient is negligible, then from (11) we have

$$\langle (A/A_0)^2 E\rangle = 0{\cdot}144(a\Omega\mathscr{B}^2\mathscr{L}/\mathscr{H}^2)\exp(2\xi/\mathscr{L})\langle [1+\zeta/\mathscr{H}]^{-4}|\partial_\tau\zeta/a\Omega|\rangle. \tag{14b}$$

The appropriate non-dimensional version of the longitudinal dispersion equation (3*c**) is

$$e^{\xi}\partial_\tau\bar{c} + \mathscr{U}_H\partial_\xi\bar{c} - \mu\partial_\xi(e^{3\xi}\partial_\xi\bar{c}) = 0. \tag{14c}$$

Here ξ, T and $\mathscr{U}_H$ are as defined in (13) and

$$\mu = 0{\cdot}144\langle [1+\zeta/\mathscr{H}]^{-4}|\partial_\tau\zeta/a\Omega|\rangle.$$

For steady flow conditions the corresponding steady solution is

$$\bar{c} = c_H + (c_M - c_H)\exp\{\tfrac{1}{3}\mu^{-1}\mathscr{U}_H[\exp(-3\xi_M) - \exp(-3\xi)]\}. \tag{14d}$$

Figure 5 (*b*) shows the results for the special case

$$c_H = 0, \quad c_M = 1, \quad \xi_M = 0, \quad \langle [1+\zeta/\mathscr{H}]^{-4}|\partial_\tau\zeta/a\Omega|\rangle = 1.$$

We note that for small discharge rates there is a very weak (logarithmic) dependence of the upstream penetration upon the fresh-water discharge.

The final example is a power-law generalization of the previous example (see figures 4*c*–*e*). The Eulerian description is

$$z/\mathscr{H} = -1 + (y/\mathscr{B})^2(x/\mathscr{L})^{-2\beta} \quad \text{with} \quad 0 \leqslant x, \quad 0 \leqslant \beta.$$

The volume upstream of a position x is given by

$$\tfrac{4}{3}\mathscr{B}\mathscr{H}\mathscr{L}[1+\zeta/\mathscr{H}]^{\frac{3}{2}}(x/\mathscr{L})^{\beta+1}(1+2\beta)^{-1}.$$

Hence it follows that

$$\left.\begin{aligned} x &= \xi[1+\zeta/\mathscr{H}]^{-\frac{3}{2}(1+\beta)}, \\ U^{(0)} &= -\tfrac{3}{2}(1+\beta)^{-1}(\xi/\mathscr{L})(\mathscr{L}/\mathscr{H})\partial_\tau\zeta[1+\zeta/\mathscr{H}]^{-(5+2\beta)/2(1+\beta)}. \end{aligned}\right\} \tag{15a}$$

We observe that the tidal velocity increases linearly with distance from the head of the estuary.

The local depth and breadth scales are

$$H = \mathscr{H}[1+\zeta/\mathscr{H}], \quad B = \mathscr{B}(\xi/\mathscr{L})^{\beta}[1+\zeta/\mathscr{H}]^{(1-2\beta)/2(1+\beta)}.$$

If the effect of the salinity gradient is negligible, then from (11) we have

$$\langle (A/A_0)^2 E\rangle = 0{\cdot}144(1+\beta)^{-1}(a\Omega\mathscr{B}^2\mathscr{L}/\mathscr{H}^2)(\xi/L)^{1+2\beta}\langle [1+\zeta/\mathscr{H}]^{(1-8\beta)/2(1+\beta)}|\partial_\tau\zeta/a\Omega|\rangle. \tag{15b}$$

The non-dimensional version of ($3c^*$) is

$$\xi^{\beta}\partial_T\bar{c}+\mathscr{U}_H\partial_\xi\bar{c}-\mu\partial_\xi(\xi^{1+3\beta}\partial_\xi\bar{c}) = 0, \tag{15c}$$

with

$$\mu = 0{\cdot}144(1+\beta)^{-1}\langle [1+\zeta/\mathscr{H}]^{(1-8\beta)/2(1+\beta)}|\partial_\tau\zeta/a\Omega|\rangle.$$

For steady flow conditions (15*c*) has the steady solution

$$\left.\begin{aligned}\bar{c} = c_H+(c_M-c_H)\exp\{\tfrac{1}{3}\mu^{-1}\beta^{-1}\mathscr{U}_H(\xi_M^{-3\beta}-\xi^{-3\beta})\} \quad \text{for} \quad \beta>0,\\ c_H+(c_M-c_H)(\xi/\xi_M)^{\mu^{-1}\mathscr{U}_H} \quad \text{for} \quad \beta=0.\end{aligned}\right\} \tag{15d}$$

Figures 5(*c*), (*d*) and (*e*) show the results for $\beta = 2, \frac{1}{2}$ and 0 with

$$c_H = 0, \quad c_M = 1, \quad \xi_M = 1, \quad \langle [1+\zeta/\mathscr{H}]^{(1-8\beta)/2(1+\beta)}|\partial_\tau\zeta/a\Omega|\rangle = 1.$$

We note that for small discharge rates the demarcation between upstream and sea conditions is approximately at the position

$$\xi = (\mathscr{U}_H/3\mu\beta)^{1/3\beta} \quad \text{for} \quad \beta > 0.$$

Thus the more the estuary resembles a channel of constant width, the greater is the sensitivity to the discharge rate. This suggests that the reclamation of valuable estuary-side land and channel dredging both contribute to the vulnerability of estuaries to variations in the river flow. An ameliorative effect ignored in the above analysis is that a reduction in shoreline irregularities can reduce E (Okubo 1973), thereby lowering the values of the discharge rates to which the above arguments apply.

It is of particular interest to compare the two solutions shown in figures 5(*a*) and (*d*), since the plans of the estuaries are identical. For large discharge rates the quantitative agreement is remarkably good. However, the agreement deteriorates rapidly with decreasing $\mathscr{U}_H$. Thus we are led to the conclusion that, even when the effects of salinity are negligible, results are transferable from one estuary to another only if there is a geometric similarity in both plan and elevation. If the salinity effect is significant, then the estuaries must also have the same estuarine Richardson number.

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